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# On extended multidimensional Schlömilch series

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**Abstract.** We extend an earlier investigation of a class of $m$-dimensional lattice sums which were phase modulated by a continuous *twist* parameter $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m)$ and involved Bessel functions $J_\nu(x)$, with argument $x = 2vq$ ($q \equiv |\boldsymbol{q}| = \sqrt{q_1^2 + \cdots + q_m^2}$; $q_j$ = integer$\forall j \in m$) such that $0 \leqslant v < \pi\tau$. This investigation—motivated partly by the Henkel–Weston conjectures and partly by the propositions of Ortner and Wagner on the solution of the hyperbolic differential operators—generalizes previous analyses to include an $m$-dimensional shift parameter $\boldsymbol{a}$ and an arbitrary scalar $b$, so that $x$ is now equal to $2v\sqrt{|\boldsymbol{q}+\boldsymbol{a}|^2 + b^2}$; at the same time, it extends the range of $v$ from 0 to $\infty$. While these generalizations are broad enough to make this a worthwhile study in its own right, the main interest here lies in applying this special technique to such diverse physical problems as finite-sized ferromagnetic or quantum fluid model systems undergoing phase transitions, the Casimir effect and topological mass generation, etc. Consequently, there arise important connections to the zeta function regularization and to heat kernel techniques.

## 1. Introduction

In response to certain conjectures made by Henkel and Weston [1,2] we reported the multidimensional Schlömilch series [3] which had emerged from our ongoing study of first- and second-order phase transitions in finite-sized (FS) systems:

$$\sum_{\boldsymbol{q}(m)}{}' \cos(2\pi\boldsymbol{\tau}\cdot\boldsymbol{q})(vq)^{-\nu} J_\nu(2vq) = -1/\Gamma(\nu+1) \tag{1}$$

$$\nu > \tfrac{1}{2}m - 1 \qquad 0 \leqslant v < \pi\tau$$

where $m (= 1, 2, 3, \ldots)$ is the dimensionality of the underlying lattice space, $J_\nu(x)$ the ordinary Bessel function, $\boldsymbol{q}(m)$ a positive vector in this space whose components $q_j$ span all integers (positive, negative or zero) such that $q \equiv |\boldsymbol{q}| > 0$, while $\boldsymbol{\tau}(m)$ is the *twist* parameter of the problem whose components $\tau_j$ are such that $0 \leqslant \tau_j \leqslant 1/2$, with $\tau \equiv |\boldsymbol{\tau}| > 0$. We note that, in view of the (symmetric) summation over $\boldsymbol{q}$, the factor $\cos(2\pi\boldsymbol{\tau}\cdot\boldsymbol{q})$ appearing in the summand of (1) may be replaced by a *phase factor* $\exp(2\pi\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{q})$, because the additional terms so introduced into the series add up identically to zero. Subsequently in our study of spin–spin correlations in an FS spherical model ferromagnet under twisted boundary conditions, we discovered a more general sum that arose in our search for an alternative representation of a sum involving modified Bessel functions, $K_\nu(2yq^*)$, analytically continued to imaginary values of the *thermogeometric* parameter $y$, so that $y^2 = -v^2 < 0$ [4]:

$$\sum_{\boldsymbol{q}(m)} \prod_{j=1}^{d^*} \cos(2\pi\tau_j q_j)(vq^*)^{-\nu} J_\nu(2vq^*) = 0 \tag{2}$$

$$q^* = \sqrt{|\boldsymbol{q}+\boldsymbol{a}|^2 + b^2} > 0 \qquad \nu > \tfrac{1}{2}m - 1 \qquad 0 \leqslant v < \pi\tau$$

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where $\boldsymbol{a}$ is an $m$-dimensional vector and $b$ an arbitrary scalar ($\geqslant 0$) such that $\varepsilon = \sqrt{a^2+b^2} > 0$ ($a = |\boldsymbol{a}|$); accordingly, the term with $\boldsymbol{q} = 0$ is included in this sum. For a proper comparison between series (1) and (2), we observe that the ($\boldsymbol{q} = 0$) term in the latter is $(v\varepsilon)^{-\nu} J_\nu(2v\varepsilon)$ which, in the limit $\varepsilon \to 0$, reduces to $1/\Gamma(\nu+1)$, making (2) consistent with (1). It is remarkable that, in the specified range of the variable $v$, the value of the series (1) and (2), with all integral $\boldsymbol{q}$ formally included, is identically zero—regardless of the values of the other parameters involved.

Almost simultaneously with us, and again in response to Henkel and Weston, Ortner and Wagner [5] reported on the evaluation of a sum which in our notation may be written as

$$\sum_{q(m)} e^{2\pi i\boldsymbol{\tau}\cdot\boldsymbol{q}} \left(v\sqrt{q^2+b^2}\right)^{-\nu} J_\nu\left(2v\sqrt{q^2+b^2}\right). \tag{3}$$

Note that this series corresponds to equation (2), with $\boldsymbol{a} \to 0$. However, while equation (2) applied only in the interval $[0, \pi\tau)$ of the variable $v$—the interval most relevant for the problem considered in [4]—the treatment of Ortner and Wagner covers the entire range of $v$ from 0 to $\infty$ that involves a discrete spectrum of singularities. Considering the potential application of these series in modelling diverse physical problems [4, 6–14], including renormalization of FS systems to examine dimensional crossovers [13, 14], and to various techniques of analysis and regularization [10–12, 15, 16], we decided to generalize the work of Ortner and Wagner by including the aforementioned *shift* parameter $\boldsymbol{a}$ which, in turn, amounts to extending our own previous results, in terms of $v$, to regions outside the interval $[0, \pi\tau)$. The approach taken for this purpose was essentially the same as the one employed in our earlier analysis of phase-modulated lattice sums [16], viz the use of the Poisson summation formula (PSF)

$$\sum_{\boldsymbol{q}} f(\boldsymbol{q}) = \sum_{\boldsymbol{\ell}} \mathcal{F}(\boldsymbol{\ell}) \tag{4}$$

where $f(\boldsymbol{q})$ is a continuous function of an $m$-dimensional vector $\boldsymbol{q}$ while $\mathcal{F}(\boldsymbol{\ell})$ is its Fourier transform

$$\mathcal{F}(\boldsymbol{\ell}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{2\pi i\boldsymbol{\ell}\cdot\boldsymbol{q}} f(\boldsymbol{q}) d^m q. \tag{5}$$

The results following from an application of this formula turn out to be quite general, involving several independent parameters and encompassing a broad class of sums and integrals of which several special cases are found in the literature (see, for instance, [17]–[22]).

## 2. Lattice sums with $\nu > \frac{1}{2}m - 1$

We begin with the generalized version of the Schlömilch series [20]

$$\mathcal{J}_{\boldsymbol{\tau}}^{a,b}(\nu|m; v) \equiv \sum_{q(m)} e^{2\pi i\boldsymbol{\tau}\cdot\boldsymbol{q}} (vq^*)^{-\nu} J_\nu(2vq^*) \tag{6}$$

extended to all $v \geqslant 0$, while other parameters are the same as defined in section 1. It is understood that, unless stated otherwise, the quantity $\sqrt{a^2+b^2} > 0$. Applying formula (4) to the $m$-dimensional sum in (6), which involves the use of the discontinuous integral of Sonine and Gegenbauer [18–20], leads to our central result, viz

$$\mathcal{J}_{\boldsymbol{\tau}}^{a,b}(\nu|m; v) = \frac{\pi^{m/2}}{v^{2\nu}} \sum_{\ell(m)}{}^{*} e^{-2\pi i\boldsymbol{a}\cdot(\boldsymbol{\ell}+\boldsymbol{\tau})} \left(\frac{v^*}{b}\right)^{\nu-m/2} J_{\nu-m/2}(2bv^*) \tag{7}$$

$$v^* = \sqrt{v^2 - \pi^2|\boldsymbol{\ell} + \boldsymbol{\tau}|^2} > 0 \qquad \nu > \tfrac{1}{2}m - 1.$$

Note that the sum $\sum^*$ in (7) is restricted to a finite set of vectors $\{\boldsymbol{\ell}\}$ that satisfy the inequality

$$|\boldsymbol{\ell} + \boldsymbol{\tau}| < v/\pi. \tag{8}$$

From this result one can see that the sum $\mathcal{J}_{\boldsymbol{\tau}}^{\boldsymbol{a},b}(\nu|m; v)$, regarded as a function of $v$, is singular whenever there exists an $\boldsymbol{\ell}$ such that $v = \pi|\boldsymbol{\ell} + \boldsymbol{\tau}|$. Some special cases of (7) are worth noting.

(i) For $\tau > 0$ and $0 \leqslant v < \pi\tau$, there is no $\boldsymbol{\ell}$ that satisfies inequality (8). The sum in question then reduces to

$$\mathcal{J}_{\boldsymbol{\tau}}^{\boldsymbol{a},b}(\nu|m; v) = 0 \tag{9}$$

independently of all parameters within the specified range. A closer examination reveals that if the *phase factor* $\exp(2\pi\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{q})$ in (6) is expanded into a sum of $2^m$ terms composed of sines and cosines, then each of the resulting sums over $\boldsymbol{q}$ vanishes on its own; one of these sums (equation (2)) was reported in [4].

(ii) For $\tau \to 0$ and $0 < v < \pi$, only $\boldsymbol{\ell} = 0$ satisfies inequality (8), giving a result that is independent of $\boldsymbol{a}$:

$$\mathcal{J}_{0}^{\boldsymbol{a},b}(\nu|m; v) = \frac{\pi^{m/2}}{b^{\nu-m/2}v^{\nu+m/2}} J_{\nu-m/2}(2bv). \tag{10}$$

(iii) In the limit $b \to 0$, equation (7) reduces to an algebraic sum

$$\mathcal{J}_{\boldsymbol{\tau}}^{\boldsymbol{a}}(\nu|m; v) = \frac{\pi^{m/2}}{\Gamma(\nu + 1 - m/2)v^{2\nu}} \sum_{\boldsymbol{\ell}(m)}^{*} \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{a}\cdot(\boldsymbol{\ell}+\boldsymbol{\tau})}(v^2 - \pi^2|\boldsymbol{\ell} + \boldsymbol{\tau}|^2)^{\nu-m/2} \tag{11}$$

in perfect agreement with the ($m = 1$) result of Kober—see equation (21.1) of Bellman [21]. When $\boldsymbol{a} = 0$ and $\nu = 1/2$, this expression corresponds to equation (13) of Ortner and Wagner [5] for general $m$. From (11) we find that

$$\mathcal{J}_{\boldsymbol{\tau}}^{\boldsymbol{a}}(\nu|m; v) = \begin{cases} 0 & \tau > 0, 0 \leqslant v < \pi\tau \qquad (12) \\ \dfrac{\pi^{m/2}}{\Gamma(\nu + 1 - m/2)v^m} & \tau = 0, 0 < v < \pi \qquad (13) \end{cases}$$

again independently of $\boldsymbol{a}$. Not surprisingly, equations (12) and (13) are special cases of equations (9) and (10), with $b \to 0$. Equation (13) plays a very useful role in the analytic continuation of FS corrections to the bulk condensate density of an ideal Bose gas embedded in a $(d-1)$-dimensional cylindrical curved space with the remaining dimension Euclidean—see, for instance, [7].

(iv) For $\boldsymbol{a} = 0$ and $b > 0$, expression (7) reproduces equation (15) of Ortner and Wagner [5] which then reduces to their equation (12) if $\nu = \frac{1}{2}$.

For further comparison with results that already appear in the literature, we consider the one-dimensional version of (7), namely

$$\sum_{q=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}\tau q} \left(\frac{v}{q^*}\right)^{\nu} J_\nu(2vq^*) = \sqrt{\pi} \sum_{\ell}^{*} \mathrm{e}^{-2\pi\mathrm{i}a(\ell+\tau)} \left(\frac{v^*}{b}\right)^{\nu-1/2} J_{\nu-1/2}(2bv^*) \tag{14}$$

$$q^* = \sqrt{(q+a)^2 + b^2} > 0 \qquad v^* = \sqrt{v^2 - \pi^2(\ell + \tau)^2} > 0 \qquad \nu > -1/2.$$

In our search of the various mathematical tables, we did not encounter any sums of the class (14) in which both $a$ and $b$ were present. Putting $a = 0$ and considering the special cases

$\tau = 0$ and $\frac{1}{2}$ (corresponding to periodic and anti-periodic boundary conditions, respectively [4]), equation (14) gives

$$\sum_{q=1}^{\infty}(\pm 1)^q \frac{1}{(q^2+b^2)^{\nu/2}} J_\nu\left(2\upsilon\sqrt{q^2+b^2}\right) = -\frac{1}{2b^\nu} J_\nu(2b\upsilon) + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \frac{\pi^{1/2}}{2b^{\nu-1/2}\upsilon^{1/2}} J_{\nu-1/2}(2b\upsilon)$$
$$\upsilon < \pi(3\pm 1)/4 \tag{15}$$

in agreement with formula 5.7.22/3 of Prudnikov *et al* [17]; furthermore, if $\nu = 0$, we recover their formula 5.7.22/1. If $a = 0$, $\tau = 0$ or 1/2 but $\nu$ is general, then we recover their formula 5.7.19/11; on the other hand, if $a = 0$ or 1/2 and $\nu = 0$ but $\upsilon$ and $\tau$ general, we recover from (14) several formulae appearing in sections 8.522–8.525 of Gradshteyn and Ryzhik [18]. In order to make a proper comparison in all these cases, the condition $\upsilon^* > 0$ has to be expressed *explicitly* in terms of the summation variable $\ell$ which, in turn, depends on the precise values of both $\upsilon$ and $\tau$. From the one-dimensional results mentioned here, we observe that many explicit formulae appearing in the various tables fall under a single class of sums represented by our general result (14).

It is instructive to see how our multidimensional sums for *different* $m$ are mutually related. For instance, we shall demonstrate here how, starting from our one-dimensional series (14), higher-dimensional series (with $m = 2, 3, \ldots$) can be obtained by iteration. For this, we make the replacement

$$b \to \sqrt{(q'+a')^2+b^2} \qquad q' = 0, \pm 1, \pm 2, \ldots \tag{16}$$

in (14), throw in the phase factor $\exp(2\pi i\tau' q')$ and sum over $q'$, with the result that

$$\sum_{q'=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} e^{2\pi i(\tau q+\tau' q')}\left(\frac{\upsilon}{q^{**}}\right)^\nu J_\nu(2\upsilon q^{**})$$
$$= \sqrt{\pi}\sum_{q'=-\infty}^{\infty} e^{2\pi i\tau' q'}\sum_{\ell}{}^{*} e^{-2\pi i a(\ell+\tau)}\left(\frac{\upsilon^*}{\sqrt{(q'+a')^2+b^2}}\right)^{\nu-1/2}$$
$$\times J_{\nu-1/2}\left(2\sqrt{(q'+a')^2+b^2}\,\upsilon^*\right) \tag{17}$$
$$q^{**} = \sqrt{(q+a)^2+(q'+a')^2+b^2} > 0 \qquad \nu > -1/2.$$

Now, the summation over $q'$ on the right-hand side of (17) can be carried out in the same manner as the summation over $q$ was carried out in (14), giving

$$\sum_{q'=-\infty}^{\infty} e^{2\pi i\tau' q'}\left(\frac{\upsilon^*}{\sqrt{(q'+a')^2+b^2}}\right)^{\nu-1/2} J_{\nu-1/2}\left(2\sqrt{(q'+a')^2+b^2}\,\upsilon^*\right)$$
$$= \sqrt{\pi}\sum_{\ell'}{}^{*} e^{-2\pi i a'(\ell'+\tau')}\left(\frac{\upsilon^{**}}{b}\right)^{\nu-1} J_{\nu-1}(2b\upsilon^{**}) \tag{18}$$
$$\upsilon^{**} = \sqrt{\upsilon^2-\pi^2(\ell+\tau)^2-\pi^2(\ell'+\tau')^2} > 0 \qquad \nu > 0.$$

Substituting (18) into (17), we obtain the two-dimensional version of (7), with $q^{**}$ and $\upsilon^{**}$ playing the roles of the $q^*$ and $\upsilon^*$ appearing there. Clearly, this process can be continued to arbitrary $m$, thereby confirming the direct approach that led to the general result (7); in the process, the restriction on $\nu$ becomes $\nu > m/2 - 1$.

## 3. Lattice sums with $\nu = \frac{1}{2}m - 1$

The extended sum (6), like its predecessors (1)–(3), converges for $\nu > m/2 - 1$ and seemingly diverges for $\nu < m/2 - 1$; for a comment on the latter case, see the last paragraph of this section. The borderline case $\nu = m/2 - 1$ presents some peculiar features that are worth noting. In order to properly observe these features, we introduce a Gaussian convergence factor $\exp(-\rho|q + a|^2)$ into the summand of (6), apply the PSF (4) to the resulting sum and subsequently let $\rho \to 0_+$. For simplicity, we set $b = 0$, obtaining an identity (valid for *all* $v \geqslant 0$), viz

$$\sum_{q(m)} \mathrm{e}^{2\pi \mathrm{i}\tau\cdot q - \rho|q+a|^2} (v|q + a|)^{-(m-2)/2} J_{(m-2)/2}(2v|q + a|)$$
$$= \frac{\pi}{\rho} \sum_{\ell(m)} \mathrm{e}^{-2\pi \mathrm{i}a\cdot(\ell+\tau) - (v^2+\pi^2|\ell+\tau|^2)/\rho} (v|\ell + \tau|)^{-(m-2)/2} I_{(m-2)/2}(2\pi v|\ell + \tau|/\rho) \tag{19}$$

where $I_\mu(x)$ is a modified Bessel function. For proceeding to the desired limit, we make use of the asymptotic formula

$$I_\mu(x) \approx \frac{\mathrm{e}^x}{\sqrt{2\pi x}} \qquad (x \gg 1). \tag{20}$$

The resulting factor in the summand describes a Gaussian distribution, $\rho^{-1/2}\exp\{-(v - \pi|\ell + \tau|)^2/\rho\}$, which then approaches the limiting form

$$\sqrt{\pi}\,\delta(v - \pi|\ell + \tau|) \tag{21}$$

where $\delta(x - x_0)$ is the Dirac delta function. Finally, we obtain the desired result:

$$\sum_{q(m)} \mathrm{e}^{2\pi \mathrm{i}\tau\cdot q} \left(\frac{v}{|q + a|}\right)^{(m-2)/2} J_{(m-2)/2}(2v|q + a|)$$
$$= \frac{\pi^{(m-2)/2}}{2v} \sum_{\ell(m)} \mathrm{e}^{-2\pi \mathrm{i}a\cdot(\ell+\tau)} \delta(|\ell + \tau| - v/\pi). \tag{22}$$

Equation (22) is remarkable, for it shows that the sum in question diverges for certain discrete values of $v$ and vanishes for all others! This derivation also shows that the sums in question may sometimes be regarded as distributions which, in some appropriate limit, assume the form of a discrete spectrum of singularities.

The case $m = 1$ of equation (22),

$$\sum_{q=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}\tau q} \cos(2v(q + a)) = \frac{1}{2} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i}a(\ell+\tau)} \delta(|\ell + \tau| - v/\pi) \tag{23}$$

is rather instructive—emphasizing 'total coherence' among the terms constituting the sum on the left whenever $v$ is of the form $\pi|\ell + \tau|$ and 'complete destructive interference' otherwise. The result for $m = 2$,

$$\sum_{q(2)} \mathrm{e}^{2\pi \mathrm{i}\tau\cdot q} J_0(2v|q + a|) = \frac{1}{2v} \sum_{\ell(2)} \mathrm{e}^{-2\pi \mathrm{i}a\cdot(\ell+\tau)} \delta(|\ell + \tau| - v/\pi) \tag{24}$$

has precisely the same property though, to begin with, one might not expect a series involving the Bessel functions $J_0(\cdots)$ to vanish for all values of the variable $v$ appearing in its argument, except for a discrete spectrum (defined through *integers* $\ell_1$ and $\ell_2$) where it diverges. The special case $a = 0$, $\tau = (\frac{1}{2}, \frac{1}{2})$ of this result was recently reported by

Grosjean [22] who claimed that the series vanishes for *all* $v$; clearly, the divergence of this series at certain characteristic values of $v$ was missed. For $m = 3$, we obtain

$$\sum_{q(3)} e^{2\pi i \tau \cdot q} \frac{\sin(2v|q+a|)}{|q+a|} = \frac{\pi}{2v} \sum_{\ell(3)} e^{-2\pi i a \cdot (\ell + \tau)} \delta(|\ell + \tau| - v/\pi) \tag{25}$$

which is consistent with the corresponding result of Ortner and Wagner [5], where $a = 0$.

The results of this section throw some light on the behaviour of extended Schlömilch series for $\nu < m/2 - 1$. Though, strictly speaking, the sum (6) is non-convergent in this range of $\nu$, successive differentiations of the left-hand side of (22) with respect to $v^2$ over a piece-wise continuous range of $v$ for which the expression on the right-hand side vanishes, show that the resulting series, for which $\nu < m/2 - 1$, would also vanish at those values of $v$. This means that in spite of appearances, when summed over all $q$, the oscillatory terms in the series would exactly cancel. At the same time, the behaviour of the sum at those discrete values of $v$ for which expression (22) diverges become even more critical at smaller $\nu$. This observation may be of some use in the analysis of the divergent series [15].

## 4. Sums containing algebraic factors

To begin with, we go back to section 2, set $b = 0$ and examine the two special cases covered by equations (12) and (13). Successive differentiations of these expressions with respect to $v^2$ yield the series

$$\sum_{q(m)} e^{2\pi i \tau \cdot q} |q+a|^{2n} \frac{J_\nu(2v|q+a|)}{(v|q+a|)^\nu}$$

$$= \begin{cases} 0 & \tau > 0,\ 0 \leqslant v < \pi\tau \qquad (26) \\ \dfrac{\pi^{m/2}\Gamma(n+m/2)}{\Gamma(m/2)\Gamma(\nu - n + 1 - m/2) v^{m+2n}} & \tau = 0,\ 0 < v < \pi. \qquad (27) \end{cases}$$

A generalization of these results, to include $b$, can be carried out with the help of the Neumann series [19]

$$\frac{J_\nu(\sqrt{z^2+h^2})}{(z^2+h^2)^{\nu/2}} = \sum_{r=0}^{\infty} \frac{J_{\nu+r}(z)}{z^{\nu+r}} \frac{(-\frac{1}{2}h^2)^r}{r!}. \tag{28}$$

For this, we replace $\nu$ in (27) by $\nu + r$, put in a factor $(-v^2b^2)^r/r!$ and sum over $r$ from 0 to $\infty$. This gives, for $\tau = 0$ and $0 < v < \pi$,

$$\sum_{q(m)} |q+a|^{2n} \frac{J_\nu(2vq^*)}{(vq^*)^\nu} = \frac{\pi^{m/2}\Gamma(n+m/2)}{\Gamma(m/2)v^{m+2n}} \sum_{r=0}^{\infty} \frac{(-v^2b^2)^r}{r!\Gamma(\nu + r - n + 1 - m/2)}$$

$$= \frac{\pi^{m/2}\Gamma(n+m/2)}{\Gamma(m/2) b^{\nu-n-m/2} v^{\nu+n+m/2}} J_{\nu-n-m/2}(2bv) \tag{29}$$

which may be compared with equation (10). The result stated in (26) remains unchanged on this generalization.

Successive integrations (rather than differentiations) with respect to $v^2$ lead to sums with algebraic factors in the denominator (rather than in the numerator). Of course, one has in this case to fix the constants of integration, which can be done by appealing to the limit $v \to 0_+$. For simplicity, we consider one-dimensional sums, with $a = b = 0$; this requires the exclusion of the ($q = 0$) term from the sum, as was done in (1). First we examine the case $\tau = 0$, with $0 < v < \pi$, to which equation (13) applies. Integrating once, we obtain

$$\sum_{q=-\infty}^{\infty}{}' \frac{1}{q^2} \frac{J_\nu(2v|q|)}{(v|q|)^\nu} = \text{constant} - \frac{2\sqrt{\pi}\, v}{\Gamma(\nu+3/2)} + \frac{v^2}{\Gamma(\nu+2)}. \tag{30}$$

In the limit $v \to 0$, this series reduces to $2\zeta(2)/\Gamma(\nu+1) = (\pi^2/3)/\Gamma(\nu+1)$, which determines the constant of integration in (30). On successive integrations, we obtain

$$\sum_{q=1}^{\infty} \frac{1}{q^{2n}} \frac{J_\nu(2vq)}{(vq)^\nu} = \frac{(-1)^n \pi v^{2n-1}}{2\Gamma(n+1/2)\Gamma(\nu+n+1/2)} + \sum_{r=0}^{n} \frac{(-1)^r}{r!} \frac{\zeta(2n-2r)}{\Gamma(\nu+r+1)} v^{2r}$$
$$n = 0, 1, 2, \ldots \qquad \nu > -2n - 1/2. \tag{31}$$

The Riemann zeta function appearing here can be expressed in a closed form through the Bernoulli numbers. A generalization of this result, using formula (28), leads to the sum

$$\sum_{q=1}^{\infty} \frac{1}{q^{2n}} \left(\frac{v}{\sqrt{q^2+b^2}}\right)^\nu J_\nu\left(2v\sqrt{q^2+b^2}\right) = \frac{(-1)^n \pi}{2\Gamma(n+1/2)} \left(\frac{v}{b}\right)^{\nu+n-1/2} J_{\nu+n-1/2}(2bv)$$
$$+ \sum_{r=0}^{n} \frac{(-1)^r}{r!} \zeta(2n-2r) \left(\frac{v}{b}\right)^{\nu+r} J_{\nu+r}(2bv). \tag{32}$$

Series with general $\tau$ can be handled in the same manner as those with $\tau = 0$; the only difficulty here is that the constants of integration are not as simple as the ones encountered above. The case $\tau = 1/2$, with $0 \leqslant v < \pi/2$, is, however, easily manageable; one obtains

$$\sum_{q=1}^{\infty} \frac{(-1)^{q+1}}{q^{2n}} \frac{J_\nu(2vq)}{(vq)^\nu} = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \frac{\eta(2n-2r)}{\Gamma(\nu+r+1)} v^{2r} \tag{33}$$

and

$$\sum_{q=1}^{\infty} \frac{(-1)^{q+1}}{q^{2n}} \left(\frac{v}{\sqrt{q^2+b^2}}\right)^\nu J_\nu\left(2v\sqrt{q^2+b^2}\right)$$
$$= \sum_{r=0}^{n} \frac{(-1)^r}{r!} \eta(2n-2r) \left(\frac{v}{b}\right)^{\nu+r} J_{\nu+r}(2bv) \tag{34}$$

where

$$\eta(s) = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \ell^{-s} = (1-2^{1-s})\zeta(s). \tag{35}$$

Several special cases of the results obtained in this section appear in sections 5.7.19–5.7.22 of Prudnikov *et al* [17]. Quite generally, our results agree with the tabulated ones which are categorized according to certain classes; most notably, the case $n = 1$ of the series (34) agrees with formula 5.7.22/4 of [17], of which there are many further special cases.

## 5. Concluding remarks

In this paper we have reported a number of results on multidimensional lattice sums containing Bessel functions with arguments that depend on several parameters and provide a significant extension of earlier conjectures [1, 2] and propositions [3–5, 22]. Our approach, which developed from an analytical study of the correlation function of a finite-sized spherical model ferromagnet [4, 6], produces new results through a *direct* application of the Poisson summation formula and is followed by a detailed verification of the special cases of these results found in the literature [17–22]. We thus generalize the so-called Schlömilch series [20] to higher dimensions ($m = 2, 3, \ldots$) and to extended values of the variable $v$, along with the inclusion of three independent parameters: (i) the *twist* parameter $\tau$, (ii) the *shift* parameter $a$ and (iii) a scalar parameter $b$ which in our original studies appeared as the magnitude of a vector $\boldsymbol{b}$ in arbitrary dimensions.

In view of the fact that we have reported several new results here and have also verified them through some of their (known) special cases, it seems appropriate to pin-point those results in this paper that are both new and significant. The most important among these (in terms of generalizing previously derived identities) are the ones given in equations (7) and (14) of section 2, which hold for $\nu > \frac{1}{2}m - 1$, followed by equations (19) and (22) of section 3, which hold for $\nu = \frac{1}{2}m - 1$. In addition, practically all the results reported in section 4, notably equations (26), (27), (29) and (31)–(34) are new.

Potentially useful applications of these series stem from their intimate connection to the zeta function regularization and to heat kernel techniques [9–12, 15] which have been used to solve, for example, FS Bose and Fermi model systems in the magnetic [4, 6, 13, 14] and fluid [7, 9] descriptions, the Casimir effect (see, for instance, [11] and references therein) and topological mass generation [8] where analytic continuation to imaginary mass values is carried out. Further applications include the possibility of applying these methodologies to FS dynamical systems near first- and second-order phase transitions in tandem with the field-theoretic renormalization group, for which there is a great deal of current interest—see, e.g. Diehl [23], Goldschmidt [24] and for more recent work Diehl and Ritschel [25].

The methodology outlined in this work can also be used to attain optimal algorithms for generating objects by way of symbolic computation (MAPLE V) that link a broad class of integrals involving certain special functions to seemingly unrelated sums (i.e. generalized zeta functions) in arbitrary dimensions through the PSF. Another useful application of the method outlined here is in the straightforward implementation of *asymptotic* properties for the various series, integrals and products by way of comparing convergence of the corresponding identities, which ordinarily may be very difficult to handle. In principle, this process can be generalized and applied indefinitely, thus generating a database which is continually updated and refined through an object oriented management system. Since there are many such avenues one can pursue in order to generalize an existing database after each generation, the user may be able to *custom fit* the program so as to best tackle the problem at hand. Preliminary work on this project, involving an extensive application to solve the O($n$) model A (purely dissipative) time-dependent Ginzburg–Landau equation by way of the Langevin formalism under non-periodic boundary conditions in the limit $n \to \infty$, is currently under way.

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